International Journal of Theoretical Physics, Vol. 38, No. 4, 1999

The Classical Regime of a Quantum Universe Obtained Through a Functional Method

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Received September 15, 1998

The functional method, introduced to deal with systems endowed with a continuous spectrum, is used to study the problem of decoherence and correlations in a simple cosmological model.

1. INTRODUCTION

One of the most important problems of theoretical physics in the recent years has been the question: How and in what circumstances does a quantum system become classical? [1]. In spite of the great effort made by physicists to find the answer, the problem is still alive [2] and we are far from a complete understanding of many of its most fundamental features. In fact the most developed and sophisticated theory on the subject, histories decoherence, is not free of strong criticisms [3].

Nevertheless there is an almost unanimous opinion that the classical regime is produced by two phenomena:

(i) *Decoherence* , which in quantum systems restores the Boolean statistic typical of quantum mechanics.

(ii) *Correlations* , which circumvent the uncertainty relation at the macroscopic level.

But the techniques to deal with these two phenomena are not yet completely developed. One of the main problems is to find a proper and unambiguous definition of the so-called *pointer basis* where decoherence takes place.

Our contribution to solve this problem is based on ideas of Segal [4] and van Howe [5], reformulated by Antoniou *et al.* [6]. We have

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0020-7748/99/0400-133 3$16.00/0

developed these ideas in refs. 7 and 8, where we have shown how the Riemann–Lebesgue theorem can be used to prove the destructive interference of the off-diagonal terms of the state density matrix yielding decoherence. Using this technique, we have found decoherence and correlations in simple quantum systems [9] where we have defined the final pointer basis in an unambiguous way.[2]

On the other hand, the appearance of a classical universe in quantum gravity models is the cosmological version of the problem we are discussing Then, decoherence and correlations must also appear in the universe [10]. In this paper, using our method, we will solve this problem in a simple quantum-cosmological model and we will find:

(i) Decoherence in all the dynamical variables and in a well-defined final pointer basis.

(ii) Correlations, in such a way that the Wigner function F_{W^*} of the asymptotic diagonal matrix ρ_* can be expanded as

$$F_{W^*}(x, p) = \int p_{\{l\}[\mathbf{a}]} F_{W\{l\}[\mathbf{a}]}(x, p)\, d\{l\}\, d[\mathbf{a}] \tag{1.1}$$

where $F_{W\{l\}[\mathbf{a}]}$ is a classical density strongly peaked[3] in a trajectory defined by the initial conditions $\mathbf{a}$ and the momenta l, and $p_{\{l\}[\mathbf{a}]}$ is the probability of each trajectory. As the limit of quantum mechanics is not classical mechanics, but classical statistical mechanics, this is our final result: The density matrix is translated into a classical density, via a Wigner function, and it is decomposed as a sum of densities peaked around all possible classical trajectories, each one of these densities weighted by its own probability.

Thus our quantum density matrix behaves in its classical limit as a statistical distribution among a set of classical trajectories. Similar results are obtained in refs. 11 and 12.

2. THE MODEL

Let us consider the flat Robertson–Walker universe [13, 14] with a metric

$$ds^2 = a^2(\eta)(d\eta^2 - dx^2 - dy^2 - dx^2) \tag{2.1}$$

where η is the conformal time and a the scale of the universe. Let us consider a free neutral scalar field and let us couple this field with the metric, with a conformal coupling ($\xi = 1/6$). The total action reads $S = S_g + S_f + S_i$ and the gravitational action is

[2] The relation of our method to the histories decoherence is studied in ref. 9. They turn out to be equivalent, but in our method the pointer basis is better defined.

[3] Precisely: peaked as allowed by the uncertainty principle.

$$S_g = M^2 \int d\eta \left[-\frac{1}{2}\dot{a}^2 - V(a) \right] \tag{2.2}$$

where M is the Planck mass, $\dot{a} = da/d\eta$, and the potential V contains the a cosmological constant term and eventually the contribution of some form of classical mater. We suppose that V has a bounded support $0 \leq a \leq a_1$. We expand the field Φ as

$$\Phi(\eta, \mathbf{x}) = \int f_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}} d\mathbf{k} \tag{2.3}$$

where the components of $\mathbf{k}$ are three continuous variables.

The Wheeler–DeWitt equation for this model reads

$$H\Psi(a, \Phi) = (h_g + h_f + h_i)\Psi(a, \Phi) = 0 \tag{2.4}$$

where

$$\begin{aligned} h_g &= \frac{1}{2M^2}\partial_a^2 + M^2 V(a) \\ h_f &= -\frac{1}{2}\int (\partial_{\mathbf{k}}^2 - k^2 f_{\mathbf{k}}^2)\, d\mathbf{k} \\ h_i &= \frac{1}{2} m^2 a^2 \int f_{\mathbf{k}}^2\, d\mathbf{k} \end{aligned} \tag{2.5}$$

with m the mass of the scalar field, $\mathbf{k}/a$ the linear momentum of the field, and $\partial_{\mathbf{k}} = \partial/\partial f_{\mathbf{k}}$.

We can now go to the semiclassical regime using the WKB method [15], writing $\Psi(a, \Phi)$ as

$$\Psi(a, \Phi) = \exp[iM^2 S(a)]\chi(a, \Phi) \tag{2.6}$$

and expanding S and X as

$$S = S_0 + M^{-1} S_1 + \ldots, \qquad \chi = \chi_0 + M^{-1}\chi_1 + \ldots \tag{2.7}$$

To satisfy Eq. (2.4) at the order M^2 the principal Jacobi function $S(a)$ must satisfy the Hamilton–Jacobi equation

$$\left(\frac{dS}{da}\right)^2 = 2V(a) \tag{2.8}$$

We can now define the (semi)classical time as a parameter $\eta = \eta(a)$ such that

$$\frac{d}{d\eta} = \frac{dS}{da}\frac{d}{da} = \pm\sqrt{2V(a)}\,\frac{d}{da} \tag{2.9}$$

The solution of this equation is $a = \pm F(\eta, C)$, where C is an arbitrary integration constant. Different values of this constant and of the $\pm$ sign give different classical solutions for the geometry.

Then, in the next order of the WKB expansion, the Schrödinger equation reads

$$i\frac{d\chi}{d\eta} = h(\eta)\chi \tag{2.10}$$

where

$$h(\eta) = h_f + h_i(a) \tag{2.11}$$

Precisely

$$h(\eta) = -\frac{1}{2}\int\left[-\frac{\partial^2}{\partial f_{\mathbf{k}}^2} + \Omega_{\mathbf{k}}^2(a) f_k^2\right] d\mathbf{k} \tag{2.12}$$

where

$$\Omega_{\varpi}^2 = m^2a^2 + k^2 = m^2a^2 + \varpi \tag{2.13}$$

where $\varpi = k^2$ and $k = |\mathbf{k}|$. So the time dependence of the Hamiltonian comes from the function $a = a(\eta)$.

Let us now consider a scale of the universe such that $a_{\text{out}} \gg a_1$. In this region the geometry is almost constant. Therefore we have an adiabatic final vacuum $|0\rangle$ and adiabatic creation and annihilation operators $a_{\mathbf{k}}^\dagger$ and $a_{\mathbf{k}}$. Then $h = h(a_{\text{out}})$ reads

$$h = \int \Omega_{\varpi} a_{\mathbf{k}}^\dagger a_{\mathbf{k}}\, d\mathbf{k} \tag{2.14}$$

We can now consider the Fock space and a basis of vectors

$$|\mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_n, \ldots\rangle \cong |\{k\}\rangle = a_{\mathbf{k}_1}^\dagger a_{\mathbf{k}_2}^\dagger \ldots a_{\mathbf{k}_n}^\dagger \ldots |0\rangle \tag{2.15}$$

where we have called $\{k\}$ the set $\mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_n, \ldots$.

The vectors of this basis are eigenvectors of h:

$$h|\{k\}\rangle = \omega|\{k\}\rangle \tag{2.16}$$

where

$$\omega = \sum_{\mathbf{k}\in\{\mathbf{k}\}} \Omega_{\varpi} = \sum_{\mathbf{k}\in\{\mathbf{k}\}} (m^2a_{\text{out}}^2 + \varpi)^{1/2} \tag{2.17}$$

We can now use this energy to label the eigenvector as

$$|\{k\}\rangle = |\omega, [\mathbf{k}]\rangle \tag{2.18}$$

where $[\mathbf{k}]$ is the remaining set of labels necessary to define the vector unambiguously. $\{|\omega, [\mathbf{k}]\rangle\}$ is obviously an orthonormal basis, so Eq. (2.14) reads

$$h = \int \omega|\omega, [\mathbf{k}]\rangle\langle\omega, [\mathbf{k}]\,| d\omega\, d[\mathbf{k}] \tag{2.19}$$

In the next section we will write this equation using a shorthand notation as

$$h = \int \omega|\omega\rangle\langle\omega|\, d\omega \tag{2.20}$$

The dynamical variables $[\mathbf{k}]$ will reappear in Section 4.

3. ENERGY DECOHERENCE

As we are dealing with a system with a continuous spectrum in ω, some care must be taken. If not, the mathematical manipulations can contain multiplication of distributions and yield infinite meaningless results. In order to deal with this problem and to always work with usual functions (not distributions) a *functional method* was introduced in refs. 7 and 8, which we will now review. The method was used to show the decoherence and the existence of correlations in ordinary quantum mechanical systems [9] and we will use it in our problem.

The physical basis of the method is the following: The states of the universe are only known and measure through a measurement process where a space of observables $\mathbb{O}$ is used. For any observable $O \in \mathbb{O}$ we can only measure the mean value of this O in a state ρ, namely

$$\langle O\rangle_\rho = Tr(\rho^\dagger O) \tag{3.1}$$

Then we can consider that the states are linear functionals over the space of observables and write

$$\langle O\rangle_\rho = \rho[O] = (\rho|O) \tag{3.2}$$

Of course, the states would be endowed with extra some properties, so we will define a convex set of observables $\mathcal{S} \subset \mathbb{O}'$, this last space being the dual of $\mathbb{O}$.

It is logical to ask that the Hamiltonian h would be contained in the space of observables $\mathbb{O}$; then the observables must be defined generalizing Eq. (2.20). This generalization, already used in refs. 7 and 8, reads

$$O = \int O_\omega |\omega\rangle\langle\omega| \, d\omega + \iint O_{\omega\omega'} |\omega\rangle\langle\omega'| \, d\omega \, d\omega'$$

$$= \int O_\omega |\omega) \, d\omega + \iint O_{\omega\omega'} |\omega,\omega') \, d\omega \, d\omega' \tag{3.3}$$

where we have introduced a *basis* $\{|\omega), |\omega, \omega')\}$ of space $\mathbb{O}$ defined as

$$|\omega) = |\omega\rangle\langle\omega|, \qquad |\omega, \omega') = |\omega\rangle\langle\omega'| \tag{3.4}$$

The terms O_ω can be considered as the (singular) diagonal terms, while the terms $O_{\omega\omega'}$ can be considered as the (regular) off-diagonal terms.

We can now define the *cobasis* $\{(\omega|, (\omega, \omega'|\}$ of space $\mathbb{O}$ (namely the basis of space $\mathbb{O}'$), which obviously satisfies the equations

$$(\omega|\omega') = \delta(\omega - \omega'), \qquad (\omega, \omega''|\omega', \omega''') = \delta(\omega - \omega') \, \delta(\omega'' - \omega''') \tag{3.5}$$

and all other $(\cdot|\cdot) = 0$.

Then if $\rho \in \mathcal{S}$, it can be expanded as

$$\rho = \int \rho_\omega(\omega| \, d\omega + \iint \rho_{\omega\omega'}(\omega, \omega'| \, d\omega \, d\omega' \tag{3.6}$$

where $\rho_\omega \geq 0$, $\rho_{\omega\omega'} = \rho^*_{\omega'\omega}$. Moreover, the ordinary functions O_ω, $O_{\omega\omega'}$, ρ_ω, and $\rho_{\omega\omega'}$ must be endowed with certain properties in order to make all the equations of the formalism well defined. These properties are listed in ref. 7 and they are assumed in this paper. Then

$$(\rho|O) = \int \rho_\omega O_\omega \, d\omega + \iint \rho_{\omega\omega'} O_{\omega'\omega} \, d\omega \, d\omega' \tag{3.7}$$

and from Eq. (2.10)

$$(\rho(\eta)|O) = \int \rho_\omega O_\omega \, d\omega + \iint \rho_{\omega\omega'} O_{\omega'\omega} e^{i(\omega - \omega')\eta} \, d\omega \, d\omega' \tag{3.8}$$

Then, when $\eta \to \infty$, essentially from the Riemann–Lebesgue theorem (see ref. 7 for details), we have

$$\lim_{\eta\to\infty} (\rho(\eta)|O) = \int \rho_\omega O_\omega \, d\omega = (\rho_*|O) \tag{3.9}$$

where

$$(\rho_*| = \int \rho_\omega(\omega| \, d\omega \tag{3.10}$$

is the equilibrium time-asymptotic state, which only contains the diagonal term. So we have proved the existence of decoherence in the energy.

4. DECOHERENCE IN THE OTHER DYNAMICAL VARIABLES

If we reintroduce the other dynamical variables in Eq. (3.10) we obtain

$$(\rho_*| = \int \rho_{\omega[\mathbf{k}][\mathbf{k}']}(\omega, [\mathbf{k}], [\mathbf{k}']| \, d\omega \, d[\mathbf{k}] \, \mathbf{d}[\mathbf{k}'] \tag{4.1}$$

where $\{(\omega, [\mathbf{k}], [\mathbf{k}']|, (\omega, \omega', [\mathbf{k}], [\mathbf{k}']\}$ is the cobasis $\{(\omega|, (\omega, \omega'|\}$, but now showing the hidden $[\mathbf{k}]$.

Let us observe that if we use polar coordinates for k, Eq. (2.3) reads

$$\Phi(x, n) = \int \sum_{lm} \phi_{klm} \, dk \tag{4.2}$$

where

$$\phi_{klm} = f_{k,l}(\eta, r) Y^l_m(\theta, \varphi) \tag{4.3}$$

where k is a continuous variable, $l = 0, 1, \ldots, ; m = -l, \ldots, l$; and Y are spherical harmonic functions. So the indices k, l, m contained in the symbol $\mathbf{k}$ are partially discrete and partially continuous.

As $\rho^\dagger_* = \rho_*$, then $\rho^*_{\bar{\omega}[\mathbf{k}'][\mathbf{k}]} = \rho_{\omega[\mathbf{k}][\mathbf{k}']}$ and therefore a set of vectors $\{|\omega, [\mathbf{l}]\rangle\}$ exists such that

$$\int \rho_{\omega[\mathbf{k}][\mathbf{k}']} |\omega, [\mathbf{l}]\rangle_{[\mathbf{k}']} \, d[\mathbf{k}'] = \rho_{\omega[\mathbf{l}]} |\omega, [\mathbf{l}]\rangle_{[\mathbf{k}]} \tag{4.4}$$

namely $\{|\omega, [\mathbf{l}]\rangle\}$ is the eigenbasis of the operator $\rho_{\omega[\mathbf{k}][\mathbf{k}']}$. Then $\rho_{\omega[\mathbf{l}]}$ can be considered as an ordinary diagonal matrix in the discrete indices like the l and the m, and a generalized diagonal matrix in the continuous indices like k.[4] Under the diagonalization process Eq. (4.1) is written as

$$(\rho_*| = \int U^{\dagger[\mathbf{l}]}_{[\mathbf{k}]} \rho_{\omega[\mathbf{l}][\mathbf{l}']} U^{[\mathbf{l}']}_{[\mathbf{k}']} U^{\dagger[\mathbf{l}'']}_{[\mathbf{k}']} (\omega, [\mathbf{l}''], [\mathbf{l}'''] | U^{[\mathbf{l}''']}_{[\mathbf{k}]} d\omega \, d[\mathbf{k}] \, \mathbf{d}[\mathbf{k}'] \, d[\mathbf{l}] \, \mathbf{d}[\mathbf{l}'] \, d[\mathbf{l}''] \, \mathbf{d}[\mathbf{l}'''] \tag{4.5}$$

where $U^{\dagger[\mathbf{l}]}_{[\mathbf{k}]}$ is the unitary matrix used to perform the diagonalization and

$$\rho_{\omega[\mathbf{l}][\mathbf{l}']} = \rho_{\omega[\mathbf{l}]} \delta_{[\mathbf{l}][\mathbf{l}']} \tag{4.6}$$

[4] E.g., We can deal with this generalized matrix rigging the space $\mathcal{S}$ and using the Gel'fand–Maurin theorem [16]; this procedure allows us to define a generalized state eigenbasis for a system with continuous spectrum. It has been used to diagonalize Hamiltonians with continuous spectra in, e.g., refs. 17–19.

where

$$\rho_{\omega[\mathbf{l}][\mathbf{l}]} = \rho_{\omega[\mathbf{l}]} = \int U_{[\mathbf{l}]}^{[\mathbf{k}]} \rho_{\omega[\mathbf{k}][\mathbf{k}']} U_{[\mathbf{l}]}^{\dagger[\mathbf{k}']}\, d[\mathbf{k}]\, d[\mathbf{k}'] \tag{4.7}$$

so we can define

$$(\omega, [\mathbf{l}]| = (\omega, [\mathbf{l}], [\mathbf{l}]| = \int U_{[\mathbf{l}]}^{[\mathbf{k}]} (\omega, [\mathbf{k}], [\mathbf{k}']|\ U_{[\mathbf{l}]}^{\dagger[\mathbf{k}']\dagger}\, d[\mathbf{k}]\, d[\mathbf{k}'] \tag{4.8}$$

We can repeat the procedure with vectors $(\omega, \omega', [\mathbf{k}], [\mathbf{k}']|$ and obtain vectors $(\omega, \omega', [\mathbf{l}]|$. In this way we obtain a diagonalized cobasis $\{(\omega, [\mathbf{l}]|, (\omega, \omega', [\mathbf{l}]\}$. So we can now write the equilibrium state as

$$\rho_* = \int \rho_{\omega[\mathbf{l}]} (\omega, [\mathbf{l}]|\ d\omega\, d[\mathbf{l}] \tag{4.9}$$

Since vectors $(\omega, [\mathbf{l}]|$ can be considered as diagonals in all the variables, we have obtained decoherence in all the dynamical variables. This fact will become clearer once we study the observables related to this vector and introduce the notion of a *final pointer basis*.

So, let us now consider the observable basis $\{|\omega, [\mathbf{l}]), |\omega, \omega', [\mathbf{l}])\}$ dual to the state cobasis $\{(\omega, [\mathbf{l}]|, (\omega, \omega', [\mathbf{l}]|\}$. From Eq. (3.4) and as the ω does not play any role in the diagonalization procedure we obtain

$$|\omega, [\mathbf{l}]) = |\omega, [\mathbf{l}]\rangle(\omega, [\mathbf{l}]|, \qquad |\omega, \omega', [\mathbf{l}]) = |\omega, [\mathbf{l}])\rangle(\omega', [\mathbf{l}]| \tag{4.10}$$

So in the basis $\{|\omega, [\mathbf{l}]), |\omega, \omega', [\mathbf{l}])\}$ the Hamiltonian reads

$$h = \int \omega |\omega, [\mathbf{l}])\, d\omega\, d[\mathbf{l}] = \int \omega |\omega, [\mathbf{l}]\rangle(\omega, [\mathbf{l}]|\ d\omega\, d[\mathbf{l}] \tag{4.11}$$

Now we can also define the operators

$$\mathbf{L} = \int \mathbf{l} |\omega, [\mathbf{l}])\, d\omega\, d[\mathbf{l}] = \int \mathbf{l} |\omega, [\mathbf{l}]\rangle(\omega, [\mathbf{l}]|\ d\omega\, d[\mathbf{l}] \tag{4.12}$$

which can also be written

$$L_i = \int l_i |\omega, [\mathbf{l}])\, d\omega\, d[\mathbf{l}] = \int l_i |\omega, [\mathbf{l}]\rangle(\omega, [\mathbf{l}]|\ d\omega\, d[\mathbf{l}] \tag{4.13}$$

where i is an index such that it covers all the dimensions of the $\mathbf{l}$. Now we can consider the set (h, L_i), which is a CSCO, since all the members of the set commute, because they share a common basis, and find the corresponding eigenbasis of the set, precise $|\omega, [\mathbf{l}]\rangle$ since[5]

[5] On some occasions we will call $h = L_0$ and $\omega = l_0$.

$$h|\omega, [\mathbf{l}]\rangle = \omega|\omega, [\mathbf{l}]\rangle \tag{4.14}$$

$$L_i|\omega, [\mathbf{l}]\rangle = l_i|\omega, [\mathbf{l}]\rangle \tag{4.15}$$

Of course the L_i are constants of the motion because they commute with h.

From all these equations we can say that:

(i) (h, L_i) is the final pointer CSCO.

(ii) $\{|\omega, [\mathbf{l}]), |\omega, \omega', [\mathbf{l}])\}$ is the final pointer observable basis.

(iii)$\{(\omega, [\mathbf{l}]|, (\omega, \omega', [\mathbf{l}]|\}$ is the final pointer states cobasis.

In fact, from Eq. (4.9) we see that the final equilibrium state has only diagonal terms in this state (those corresponding to vectors $(\omega, [\mathbf{l}]|$), it has no off-diagonal terms {those corresponding to vectors $(\omega, \omega', [\mathbf{l}]|$, $(\omega, [\mathbf{k}], [\mathbf{k}']|$, or $(\omega, \omega', [\mathbf{k}], [\mathbf{k}']|\}$, and therefore we have decoherence in all the dynamical variables.

5. CORRELATIONS

As was explained in ref. 9, correlations are computed in the limit of small $\hbar$. Under this assumption it is demonstrated that for each observable (e.g., momentum or energy) we can find a canonically conjugate dynamical variable (e.g., configuration variables or the hand of a clock, namely time) if we neglect $O(\hbar)$ terms. So we will use these approximate canonically conjugated variables in this section, since, we repeat, we are only interested in observational conditions where $\hbar$ can be considered very small.

In accordance with this idea the canonically conjugate variable of h would be essentially η, but since ρ_* is an η constant, the time variable is completely unimportant in this section (we will discuss this matter further in Section 6.3). Let us call a_i the canonically conjugate variable [precisely, "conjugate variable up to $O(\hbar)$ terms"] of the observable L_i. Then (a_i) will be our configuration variables and (L_i) our momentum variables. [We will call x, p the old variables of Eq. (2.1) and ξ, $\mathbf{a}$ the new configuration variables and π, $\mathbf{l}$ the the new momentum variables]. Using these new variables, we will compute the Wigner function [20] corresponding to the operator ρ_*. We can also use the usual transcription rules:

$$h \to i\frac{\partial}{\partial \eta}, \qquad L_i \to -i\frac{\partial}{\partial a_i} \tag{5.1}$$

since the difference with respect to other transcription rules in other coordinates is just $O(\hbar)$. Then

$$\langle \eta, [\Delta\mathbf{a} + \mathbf{a}_0]|\omega[\mathbf{l}]\rangle = e^{i(-\omega\eta + \mathbf{l}\cdot\Delta\mathbf{a})}\langle 0, [\mathbf{a}_0]|\omega[\mathbf{l}]\rangle \tag{5.2}$$

but we will only consider the state of affairs for $\eta = 0$. We will call

$$[\mathbf{a}] = [\Delta\mathbf{a} + \mathbf{a}_0], \qquad \{l\} = (\omega, [\mathbf{l}]), \qquad \{\pi\} = (\omega, [\pi]) \tag{5.3}$$

where in the second and third equations we have restored the notation of Eq. (2.15) With this notation and for $\eta = 0$, eq. (5.2) reads

$$\langle [\mathbf{a}] | \{l\} \rangle = e^{i\Delta\mathbf{a}\cdot\mathbf{l}} \langle [\mathbf{a}] | \{l\} \rangle \tag{5.4}$$

The Wigner function corresponding to matrix ρ_* reads

$$F_{W*}(x, p) = F_{W*}(\xi, \pi) \sim \int_{-\infty}^{\infty} \langle \xi - \eta | \rho_* | \xi + \eta \rangle \, e^{2i\pi\cdot\eta} d[\eta] \tag{5.5}$$

Then from Eqs. (4.9) and (5.4) [and Eq. (6.7), written for the spatial coordinates for the continuous indices; see details in ref. 9] we have

$$\begin{aligned} F_{W*}(x, p) &\sim \int \ldots \int_{-\infty}^{\infty} d[\eta] \int d\{l\} \, \rho_{\{l\}} \langle \xi - \eta | (\{l\} \| \, \xi + \eta \rangle e^{2i\pi\cdot\eta} \\ &= \int \ldots \int_{-\infty}^{\infty} d[\eta] \int d\{l\} \, \rho_{\{l\}} e^{i(\xi-\eta)l} (\{l\} | [\mathbf{a}_0]) e^{-i(\xi+\eta)l} \, e^{2i\pi\cdot\eta} \\ &\sim \int d\{l\} \, \rho_{\{l\}} | \langle \{l\} | [\mathbf{a}_0] \rangle |^2 \delta([\pi] - [\mathbf{l}]) \end{aligned} \tag{5.6}$$

where the probability $(\{\boldsymbol{l}\}|[\mathbf{a}_0])$ has been written with the more familiar (but not rigorous) symbol $|\langle \{l\} | [\mathbf{a}_0] \rangle|^2$ (which turns out to be rigorous only for discrete l).[6] The $\delta([\pi] - [\mathbf{l}])$ does not contain the energy. But in the footnote of Section 6.3 we will prove that a δ-term in the energy can also be added, so finally:

$$\begin{aligned} F_{W*}(x, p) &\sim \int d\{l\} \, \rho_{\{l\}} | \langle \{l\} | [\mathbf{a}_0] \rangle |^2 \, \delta(\{\pi\} - \{l\}) \\ &= \int d\{l\} \, \rho_{\{l\}} | \langle \{l\} | [\mathbf{a}_0] \rangle |^2 \prod_{i=0} \delta(\pi_i - l_i) \end{aligned} \tag{5.7}$$

The last equation can be interpreted as follows:

(i) $\delta(\{p\} - \{l\})$ is a classical density function, strongly peaked at certain values of the constants of motion $\{l\}$, corresponding to a set of trajectories

[6] We see that ξ disappears from the equation, so $F_{W*}(\xi, \pi)$ is neither a function of the position ξ nor of the conventional origin $\mathbf{a}_0$. This is a consequence of the spatial homogeneity of the model we are studying. Moreover, it can also be seen that if we use Eq. (5.2) with $\eta \neq 0$ the function $F_{W*}(\xi, \pi)$ is a constant of η (as it should be). So, essentially, in all this section we are dealing with functions that are constants in time.

where the momenta are equal to the eigenvalues of Eqs. (4.14) and (4.15), namely $\pi_i = l_i$ ($i = 0, 1, 2, \ldots$). This fact already shows the presence of correlations in our model. In fact, we can consider each set of trajectories labeled by $\{l\}$ (i.e., a "history" obtained using some apparatus that measure only the momenta) and prove that in these trajectories the usual coordinate x and the usual momentum p are correlated as allowed by the uncertainty principle [9]. For the conjugate variables $\mathbf{l}$ and $\mathbf{a}$, $\mathbf{l}$ is completely defined and $\mathbf{a}$ is completely undefined, also satisfying the uncertainty principle.

(ii) $\rho_{\{l\}}$ is the probability to be in one of these sets of trajectories labeled by $\{l\}$. Precisely, if some initial density matrix is given, from Eq. (4.9) it is evident that its diagonal terms $\rho_{\{l\}}$ are the probabilities to be in the states $(\omega, [\mathbf{l}]|$ and therefore the probability to find, in the corresponding classical equilibrium density function $F_{W*}(x, p)$, the density function $\delta(\{p\} - \{l\})$, namely the set of trajectories labeled by $\{l\} = (\omega, [\mathbf{l}])$.

(iii) The factor $|\langle[\mathbf{a}_0]|\{l\}\rangle|^2$ corresponds to the probability that one of the trajectories $\{l\}$ would pass by $\mathbf{a}_0$ at time $\eta = 0$ and it can easily be computed from the model.[7]

(iv) Therefore $\rho_{\{l\}}|\langle[\mathbf{a}_0]|\{l\}\rangle|^2 = p_{\{l\}[\mathbf{a}_0]}$ is the probability that, given an initial density matrix, a trajectory with constant of the motion $\{l\}$ would pass by the point $\mathbf{a}_0$ at time $\eta = 0$ and then would follow the classical trajectory:

$$\mathbf{a} = \mathbf{l}\eta + \mathbf{a}_0 \tag{5.8}$$

But, $p_{\{l\}[\mathbf{a}_0]}$ is not a really function of $\mathbf{a}_0$, it is simply a constant in $\mathbf{a}_0$ (as explained in a previous footnote); since this is only an arbitrary point and our model is spatially homogenous, we can write

$$p_{\{l\}[\mathbf{a}_0]} = \int p_{\{l\}[\mathbf{a}_0]} \prod_{i=1} \delta(\xi_i - a_{0i})\, d[\mathbf{a}_0] \tag{5.9}$$

In this way we have changed the role of $\mathbf{a}_0$; it was a fixed (but arbitrary) point and it is now a variable that moves all over the space. Then Eq. (5.7) reads

$$F_{W*}(x, p) \sim \int p_{\{l\}[\mathbf{a}_0]} \prod_{i=0} \delta(\pi_i - l_i) \prod_{j=1} \delta(\xi_j - a_{0j})\, d[\mathbf{a}_0]\, d\{l\} \tag{5.10}$$

So if we call

$$F_{W\{l\}[\mathbf{a}_0]}(x, p) = \prod_{i=0} \delta(\pi_i - l_i) \prod_{j=1} \delta(\xi_j - a_{0j}) \tag{5.11}$$

[7] From the spatial homogeneity of the problem and the usual normalization we have $(\{l\}|[\mathbf{a}_0]) = |\langle[\mathbf{a}_0]|\{l\}\rangle|^2 \sim \omega^{-n}$, with n the particle number.

we have

$$F_{W*}(x, p) \sim \int p_{\{l\}[\mathbf{a}_0]} F_{W\{l\}[\mathbf{a}_0]}(x, p)\, d[\mathbf{a}_0]\, d\{l\} \tag{5.12}$$

From Eq. (5.11) we see that $F_{W\{l\}[\mathbf{a}_0]}(x, p) \neq 0$ only in a narrow strip around the classical trajectory (5.8) defined by the momenta $\{l\}$ and passing through the point $[\mathbf{a}_0]$ [the density function is actually peaked, as allowed by the uncertainty principle, so its width is essentially $O(\hbar)$ since the δ-functions of all the equation are really Dirac deltas when $\hbar \to 0$]. These trajectories explicitly show the presence of correlations in our model.[8] So we have proved Eq. (5.12) which, in fact, is Eq. (1.1), as announced.[9]

Then we have obtained the classical limit. When $\eta \to \infty$ the quantum density ρ becomes a diagonal density matrix ρ_*. The corresponding classical distribution $F_{w*}(x, p)$ can be expanded as a sum of classical trajectory density functions $F_{W\{l\}[\mathbf{a}_0]}(x, p)$, each weighted by its corresponding probability $p_{\{l\}[\mathbf{a}_0]}$. So, as the limit of our quantum model we have obtained a statistical classical mechanical model, and the classical realm appears.

6. DISCUSSION AND COMMENTS

6.1. Characteristic Times

The decaying term of Eq. (3.8) (i.e., the second term of the r.h.s.) can be analytically continued using the techniques explained in refs. 7, 14, and 19. In these papers it is shown that each pole $z_i = \omega_i - i\gamma_i$ of the S-matrix of the problem considered leads to a damping factor $e^{-\gamma_i \eta}$. Then if $\gamma = \min(\gamma_i)$ the characteristic decoherence time is γ^{-1}. This computation is done in the specific models of refs. 14. If $\gamma \ll 1$, even if the Riemann–Lebesgue theorem is always valid, there is no practical decoherence since $\gamma^{-1} \gg 1$.

[8] Of course, our "trajectories" are not only one trajectory for a one-particle state, but they are n trajectories (each one corresponding to momenta $(l_1, l_2, \ldots, l_n) = \{l\}$ and passing by a point $(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n) = [\mathbf{a}]$) for the n-particle states. As $p_{\{l\}\{a\}} \sim \omega^{-n}$, the probability decreases with the particle number and the energy.

[9] In this section we have faced the following problem: $F_{W*}(\xi, \pi)$ is a ξ constant that we want to decompose into functions $F_{W\{l\}[\mathbf{a}_0]}(x, p)$ which are different from zero only around the trajectory (5.8) and therefore are variables in ξ. Then, essentially we use the fact that if $f(x, y) = g(y)$ is a constant function in x, we can decompose it as

$$g(y) = \int g(y)\delta(x - x_0)\, dx_0$$

namely the densities $\delta(x - x_0)$ are peaked in the trajectories $x = x_0 = \text{const.}$, $y = \text{var.}$, and, therefore, are functions of x. These trajectories play the role of those of Eq. (5.9). As all the physics, including the correlations, is already contained in Eq. (5.7) [as explained in point (i)] the reader may just consider the final part of this section, from Eq. (5.9) to Eq. (5.12), a didactic trick.

6.2. Decoherence of Sets of Trajectories

It is usual to say that in the classical regime there is decoherence of the set trajectories labeled by the constants of motion $\omega, [\mathbf{I}]$. This result can easily be obtained with our method in the following way.

(i) Let us consider two different states $|\omega[\mathbf{I}]\rangle$ and $|\omega'[\mathbf{I}']\rangle$ that will define classes of trajectories with different constants of motion $(\omega, [\mathbf{I}]) \neq (\omega', [\mathbf{I}'])$. We must compute

$$\langle\omega[\mathbf{I}]|\rho_*|\omega'[\mathbf{I}']\rangle = (\rho_*\|\omega\omega'[\mathbf{I}][\mathbf{I}']) = \left[\int \rho_{\omega''[\mathbf{I}'']}(\omega''[\mathbf{I}''] |\, d\omega''\, d[\mathbf{I}'']\right] |\omega\omega'[\mathbf{I}][\mathbf{I}']) = 0 \tag{6.1}$$

due to the orthogonality of the basis $\{(\omega, [\mathbf{I}]|, (\omega, \omega', [\mathbf{I}]|\}$.

(ii) But if we compute

$$\langle\omega[\mathbf{I}]|\rho_*|\omega[\mathbf{I}]\rangle = (\rho_*\|\omega[\mathbf{I}]) = \left[\int \rho_{\omega''[\mathbf{I}'']}(\omega''[\mathbf{I}''] |\, d\omega''\, d[\mathbf{I}'']\right] |\omega\omega[\mathbf{I}])$$

$$= \int \rho_{\omega''[\mathbf{I}'']}\, \delta(\omega - \omega'')\, \delta([\mathbf{I}] - [\mathbf{I}'])\, d\omega''\, d[\mathbf{I}''] = \rho_{\omega[\mathbf{I}]} \neq 0 \tag{6.2}$$

The last two equations complete the demonstration. We will discuss the problem of the decoherence of two trajectories with the same $\{I\}$ but different $[\mathbf{a}_0]$ in Section 6.4.

6.3. A Discussion of Time Decoherence

It is well known that one of the main problems of quantum gravity is the problem of the time definition [21]. A poorly studied feature of this problem is that there must be a decoherence process related to time, since time is as a classical variable. In this subsection, using the functional technique, we will give a model that shows that this is the case (but we must emphasize that this subject is not completely developed).

We must compute $\langle\eta|\rho_*|\eta'\rangle$, where $|\eta\rangle$ and $|\eta'\rangle$ are two states of the system for different times that evolve as[10]

$$|\eta\rangle = e^{-i\hbar\nu}|0\rangle \tag{6.3}$$

$|\eta\rangle\langle\eta'|$ can be considered as an observable; then

$$\langle\eta'|\rho_*|\eta\rangle = (\rho_*\|\eta\rangle\langle\eta'|) \tag{6.4}$$

[10] Cf. Eq. (5.2) and remember that therefore in this subsection we are dealing with equations that are only valid when $\hbar \to 0$.

But

$$(\omega\|\eta\rangle\langle\eta'|) = (\omega|e^{-ih\nu}|0\rangle\langle 0|e^{ih\eta'}) = [e^{ih\nu'}(\omega|e^{-ih\eta}]\|0\rangle\langle 0|) \tag{6.5}$$

Now, for any observable O we have

$$\begin{aligned}
&[e^{ih\eta'}(\omega|e^{-ih\eta}]\|O) \\
&= [e^{ih\eta'}(\omega|e^{-ih\eta}]|\left[\int O_{\omega'}|\omega')\,d\omega' + \iint O_{\omega'\omega''}|\omega', \omega'')\,d\omega'\,d\omega''\right] \\
&= [e^{ih\eta'}(\omega|e^{-ih\eta}]|\left[\int O_{\omega'}|\omega')\,d\omega' + \ldots\right] \\
&= (\omega|\left[\int O_{\omega'}\,e^{-i\omega'\eta}|\omega')e^{i\omega'\eta'}\,d\omega'\right] \\
&= e^{-i\omega(\eta'-\eta)}(\omega|O)
\end{aligned} \tag{6.6}$$

Thus[11]

$$(\omega\|\eta\rangle\langle\eta'|) = e^{-i\omega(\eta'-\eta)}(\omega\|0\rangle\langle 0|) \tag{6.7}$$

So now we can compute the following two cases: (i)

$$\begin{aligned}
\langle\eta'|\rho_*|\eta\rangle = (\rho_*\|\eta\rangle\langle\eta'|) &= \left[\int \rho_\omega(\omega|d\omega\right]\|\eta\rangle\langle\eta'|) \\
&= \int \rho_\omega e^{-i\omega(\eta'-\eta)}(\omega\|0\rangle\langle 0|)\,d\omega \to 0
\end{aligned} \tag{6.8}$$

when $|\eta' - \eta| \to \infty$, due to the Riemann–Lebesgue theorem.

(ii) Analogously,

$$\langle\eta|\rho_*|\eta\rangle = \int \rho_\omega(\omega\|0\rangle\langle 0|)\,d\omega \neq 0 \tag{6.9}$$

So we have time decoherence for two times η and η' if they are far enough apart.

This result is important for the problem of time definition, since in order to have a reasonable classical time this variable must first decohere. The result above shows that this is the case for η and η' far enough apart, but also that, for closer times (namely, such that their difference is smaller than the Planck time) there is no decoherence and time cannot be considered as

[11] Considering this equation and repeating the procedure done from Eq. (5.5) to Eq. (5.7), we can see that there is an extra δ-factor $\delta(\pi_0 - l_0)$ related to the energy. Therefore the trajectories described in Section 5 conserve, not only the momenta l, but also the energy h.

a classical variable. Classical time is a familiar concept, but the real nature of the nondecohered quantum time is open to discussion.

6.4. Decoherence in the Space Variables

Now that we know that there is time decoherence we can repeat the reasoning for the rest of the variables ξ at time $\eta = 0$, and change Eq. (6.3) by

$$|\xi\rangle = e^{i\xi\cdot\mathbf{1}}|\mathbf{0}\rangle \tag{6.10}$$

and we will reach the following conclusions:

(i)

$$\langle\xi|\rho_*|\xi'\rangle \to 0 \tag{6.11}$$

when $|\xi - \xi'| \to \infty$.

(ii)

$$\langle\xi|\rho_*|\xi\rangle \neq 0 \tag{6.12}$$

Therefore there is also decoherence between two trajectories with the same $\{l\}$ but different $[\mathbf{a}_0]$.

These facts complete the scenario about decoherence and correlations.

7. CONCLUSION

After the WKB expansion and the decoherence and correlation processes our quantum model has the following features.

(i) It has a defined classical time η and a defined classical geometry related by Eq. (2.10).

(ii) Decoherence has appeared in a well-defined final pointer basis.

(iii) The quantum field has led to a classical final distribution function [Eq. (5.12)] that is a weighted average of some set densities, each one related to a classical trajectory. The weight coefficients are the probabilities of each trajectory.

We can foresee that if instead of a spinless field we would couple the geometry with a spin-2 metric fluctuation field the result would be more or less the same. Then the corresponding quantum fluctuations would become classical fluctuations that would correspond to matter inhomogeneities (galaxies, clusters of galaxies, etc.) that will move along the trajectories described above. This subject will be treated elsewhere.

ACKNOWLEDGMENTS

This work was partially supported by grants CI1*-CT94-0004 and PSS*-0992 of the European Community, PID 3183/93 of CONICET, and EX053

of the Buenos Aires University, as well as grants from the Fundación Antorchas and the OLAM Foundation.

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